MATH 147, FALL 2024: FINAL EXAM PRACTICE PROBLEMS

Below are problems to practice for the final exam. The problems below, together with the problems from the three midterm exams, are a good representation of what to expect on the final exam. There will also be a few short answer questions on the final exam.

1. Let $f(x,y) = \begin{cases} x^2+y^2, & \text{if } x^2+y^2<1 \\ 1, & \text{if } x^2+y^2 \geq 1. \end{cases}$ Determine at which points $f(x,y)$ is continuous.

2. Show that the function $f(x,y) = \begin{cases} \frac{2^x-1)(\sin(y))}{xy}, \text{if } xy \neq 0 \\ \ln(2), \text{if } xy = 0 \end{cases}$ is continuous at (0,0).

3. Use the limit definition to show that $f(x,y) = 5x+4y^2$ is differentiable at (2,1).

4. From class, we saw that if the first order partial derivatives of $f(x,y)$ are continuous in a neighborhood of (a,b), then $f(x,y)$ is differentiable at (a,b). This problem shows why those conditions are necessary. Let

$$f(x,y) = \begin{cases} \frac{2xy(x+y)}{x^2+y^2}, \text{if } (x,y) \neq (0,0) \\ 0, \text{if } (x,y) = (0,0) \end{cases}.$$

Show that:

(i) $f(x,y)$ is continuous at $(0,0)$.
(ii) Use the limit definitions to show that $f_x(0,0)$ and $f_y(0,0)$ exist and are equal to 0.
(iii) Conclude that $L(x,y) = 0$.
(iv) Show that $f(x,y)$ is not differentiable at (0,0).
(v) Show that $f_x(x,y)$ is not continuous at (0,0).

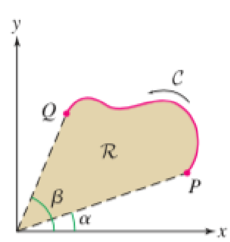
5. Find and classify the critical points for $f(x,y) = x^4 - 4xy + 2y^2$.

6. Find the absolute maximum and absolute minimum values of $f(x,y) = x^2y$ on the closed and bounded set $D: 0 \leq 4x^2 + 9y^2 \leq 36$.

7. Let S be the surface parametrized by $G(u,v) = (2u\sin(\frac{v}{2}), 2u\cos(\frac{v}{2}), 3v)$, with $0 \leq 1$ and $0 \leq v \leq 2\pi$.

(i) Find the tangent plane to S at the point $P = G(1, \frac{\pi}{3})$.
(ii) Find the surface area of S.

8. Let C be a curve from the point P to the point Q in the xy-plane. Let $\mathcal{R}$ be the region enclosed by C and the two radial lines from the origin to P and Q. (See the figure below.) Use Green's Theorem to show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ gives the area of $\mathcal{R}$, for $\mathbf{F} = -\frac{y}{2}\vec{i} + \frac{x}{2}\vec{j}$.



9. Let C be the triangle with vertices (1,0,0), (0,2,0), (0,0,1). Compute $\int_C \mathbf{F} \cdot \, d\mathbf{r}$, for the vector field $\mathbf{F} = (x^2+yz, x+y, y-z^2)$.

10. Let $f(x,y) = \sqrt{|xy|}$. Write out details showing:

(a) $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist.

(b) $f(x, y)$ is not differentiable at (0,0).
(c) Part (b) does not contradict part (a).

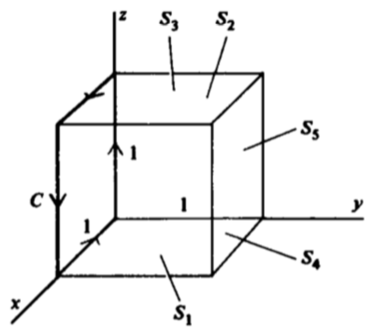
11. Evaluate $\int\int_S \text{Curl}\mathbf{F} \cdot d\mathbf{S}$, for $\mathbf{F} = (-y + z\sin(x), x, z^3)$ and S the surface defined by the equation $x^2 + \frac{y^2}{4} + z^2 + z^4x^2 = 1$, with $z \geq 0$.

12. Verify the Divergence Theorem for $\mathbf{F} = (-x^2, y^2, -z^2)$ and S rectangular box $[0, 3] \times [-1, 2] \times [1, 2]$.

13. Let $\mathbf{F} = (z^2, x^2, -y^2)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the path traversing counterclockwise the square with sides of length s centered at $(x_0, y_0, 0)$. Then divide this number by the area of the square and take the limit as $s \to 0$. Compare this with $(\text{Curl } \mathbf{F})(x_0, y_0, 0) \cdot k$.

14. Let C be the curve obtained by intersecting the cylinder $x^2 + y^2 = 1$ with the plane $x + y + z = 1$, and $\mathbf{F} = -y^3\vec{i} + x^3\vec{j} + -z^3\vec{k}$. Set up the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ as a single intgeral over an interval of the form $[a, b]$. Now evaluate this line integral by using Stoke's Theorem.

15. Verify Stoke's Theorem for $\mathbf{F} = (z^2, -y^2, 0)$ and C the square of side 1 oriented as shown, lying in the xz-plane and S the open box with sides S_1, S_2, S_3, S_4, S_5. What happens, if instead, you take S to be the square enclosed by C?



16. Calculate, without using Stoke's Theorem, $\int\int_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S}$, for $\mathbf{F} = (3y^2 + 2y)\vec{i} + 3z^2\vec{j} + 3x^2\vec{k}$ and S_1 the inverted cone $z = 1 - \sqrt{x^2 + y^2}$, with vertex $(0, 0, 1)$, and $z \geq 0$. Then calculate directly $\int\int_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S}$, for S_2 the unit disk in the xy-plane. The answers you get should be the same. This shows the consequence of Stoke's Theorem, that surfaces integrals of the curl of a vector field over surfaces sharing the same boundary are independent of the surface.